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DOUBLE ENDED QUEUEING SYSTEM WITH DISCRETE TIME

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Abstract

In this paper, an attempt has been made to obtain closed form transient solution for the double ended queuing system n discrete time. It is also further shown how the corresponding results in continuous time can be obtained.

Keywords: *Double Ended Queuing System, Discrete Time*

1. Introduction

The concept of double ended queue was introduced by Kendall (1951). The example of such a queuing system is a taxi-stand where at times passengers queue up for taxis and at other times taxis wait for passengers. Shrivastava and Kashyap (1982) obtained the transient solution of a double ended, queue in terms of summation of the integrals of Bessel functions, which is quite cumbersome and difficult to compute. Sharma (1990) provided a simple algebraic closed from expression and easy to compute for the transient probabilities in continuous time. In this paper, an attempt has been made to obtain closed form transient solution for the double ended queuing system in discrete time. It is also further shown how the corresponding results in continuous time can be obtained.

2. Assumptions

(1) The numbers on the negative axis stand for taxis waiting for customers or passengers where as the numbers on the positive axis denote the passengers waiting for taxis.

(2) The queue consists of finite waiting space.

(3) Probabilities of passengers arrival is □ and probabilities of taxis arrival is □ .

(4) Arrival probabilities of passengers and taxis follows geometric distributions.

(5) The queue discipline is FCFS.

3. Notations

X_k	:	Number of customers or taxis at epoch k.
M, N	:	Maximum number of taxis and customers respectively

□□ □ □ □ : Arrival probabilities of customers and taxis respectively

ϕ : □ (l– □)

ψ : □ (1– □)

4. Analysis Of The Model

Let X_m be either the number of customers or taxis at discrete time epoch m then $\{X_k\}$, $k \geq 0$ is an integer valued discrete stochastic process.

Taking values –M, –M+1, ... 0, 1, 2, ... N – 1, N; $X_k = k$ $(-M \leq k \leq N)$ implies that there are either M taxis or N customers waiting at epoch k. On each arrival or service the process X_k behaves as a discrete time Markov – Process and represents the state of the system.

Let $P_m(n)$ denote the probability that the system is in the n^{th} state at the beginning of the m^{th} epoch.

The difference – differential equations may be written as

$P_{m+1}(-M) = P_m(-M)(1-\square) + P_m(-(M-1))\square(1-\square)$ ....(1)

$P_{m+1}[-(M-1)] = P_m[-(M-1)][(1-\square)(1-\square) + \square\square] + P_m[-(M-2)\square(1-\square)] + P_m(-M)\square$ ....(2)

$P_{m+1}(k) = P_m(k)[(1-\square)(1-\square) + \square\square] + P_m(k+1)\square(1-\square)] + P_m(k-1)\square(1-\square); -(M-2) \leq k \leq N-2$ ....(3)

$P_{m+1}(N-1) = P_m(N-1)[(1-\square)(1-\square) + \square\square] + P_m(N)\square\square + P_m(N-2)\square(1-\square)]$ ....(4)

$P_{m+1}(N) = P_m(N)(1-\square) + P_m(N-1)\square(1-\square)$ ….(5)

With $P_o(-M) = 1$, $P_o(i) = 0$; $-(M-1) \leq i \leq N$

Let $P_z(n)$ be the probability generating function (p.g.f.) of $P_m(n)$ defined as

$$P_Z(n) = \sum_{m=0}^{\infty} z^m P_m(n)\ ;\ |z| \leq 1$$

Now taking the p.g.f. of difference-differential equations we have

$$s P_z(-M) - m(1-l) Pz(-(M-1)) = \frac{1}{z} \qquad ....(6)$$

$$-\square P_z(-M) + (s+\phi+\psi) P_z[-(M-1)] - \phi P_z[-(M-2)] = 0 \qquad ....(7)$$

$$-\phi P_z(k-1) + (s+\phi+\psi) P_z(k) - \psi P_z(k+1) = 0 \qquad ....(8)$$

$$-\phi P_z(N-2) + (s+\phi+\psi) P_z(N-1) - \square P_z(N) = 0 \qquad ....(9)$$

$$-\phi P_z(N-1) + (s+\square) P_z(N) = 0 \qquad ….(10)$$

$$AP \quad = (\delta_{o(-M)}, \delta_{o(-M+1)} \ldots . \delta_{oo}, \ \delta_{o1, \ldots} \delta_{oN}) \qquad \ldots .(11)$$

Where A is a real tri-diagonal matrix of order (N–k+2) × (N–k+2) ; P is a column vector and δ_{ok} the Kroneeker delta defined as

$$\delta_{ok} = \begin{cases} \dfrac{1}{z}; \ k = -M \\ 0 \ ; \text{Otherwise} \end{cases}$$

$$A(S) = \begin{array}{c|cccccccccccc|}
 & -M & -M+1 & -M+2 & -M+3 & \ldots & -1 & 0 & 1 & \ldots & N-3 & N-2 & N-1 & N \\
\hline
-M & s+\square & -\psi & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
-M+1 & -\square & s+\phi+\psi & -\psi & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
-M+2 & 0 & -\phi & s+\phi+\psi & -\psi & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
. & . & . & . & . & \ldots & . & . & . & \ldots & . & . & . & . \\
. & . & . & . & . & \ldots & . & . & . & \ldots & . & . & . & . \\
. & . & . & . & . & \ldots & . & . & . & \ldots & . & . & . & . \\
-1 & 0 & 0 & 0 & 0 & \ldots & -\phi & s+\phi+\psi & -\psi & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & -\phi & s+\phi+\psi & \ldots & -\psi & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & -\phi & \ldots & s+\phi+\psi & 0 & 0 & 0 \\
. & . & . & . & . & \ldots & . & . & . & \ldots & . & . & . & . \\
. & . & . & . & . & \ldots & . & . & . & \ldots & . & . & . & .
\end{array}$$

$$
\begin{array}{c}
. \\
N-2 \\
N-1 \\
N
\end{array}
\left|\begin{array}{cccccccccccccc}
. & . & . & . & \cdots & . & . & . & \cdots & . & . & . & . \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & s+\phi+\psi & -\psi & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -\phi & s+\phi+\psi & -\square \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & -\phi & s+\square
\end{array}\right|
$$

$(M + N + 1) \times (M + N + 1)$

$$
\text{And} \qquad P = \left|\begin{array}{c}
P_z(-M) \\
P_z(-M+1) \\
P_z(-M+2) \\
. \\
. \\
. \\
P_z(-1) \\
P_z(0) \\
P_z(1) \\
. \\
. \\
. \\
. \\
. \\
P_z(N-2) \\
P_z(N-1) \\
P_z(N)
\end{array}\right|_{(N+M+1) \times 1}
$$

Using Cramer's rule

$$
P_z(N) = \frac{\left|A_{N+M+1}(s)\right|}{\left|A(s)\right|}; \ |z| \le 1
$$

Where $A_{N+M+1}(s)$ is obtained from matrix A by replacing the $(N+M+1)^{th}$ column by right hand side of (11) and $|A(s)|$ is the determinant of A (s).

$$
D(s) = \begin{array}{c|cccccccccc|}
 & -M+1 & -M+2 & \ldots & -1 & 0 & 1 & \ldots & N-2 & N-1 & N \\
-M+1 & s+\phi+\psi & -\sqrt{\lambda\phi} & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
-M+2 & -\sqrt{\lambda\psi} & s+\phi+\psi & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
. & . & . & \ldots & . & . & . & \ldots & . & . & . \\
. & . & . & \ldots & . & . & . & \ldots & . & . & . \\
. & . & . & \ldots & . & . & . & \ldots & . & . & . \\
-1 & 0 & 0 & \ldots & s+\phi+\psi & -\sqrt{\phi\psi} & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & -\sqrt{\phi\psi} & s+\phi+\psi & -\sqrt{\phi\psi} & \ldots & -\psi & 0 & 0 \\
1 & 0 & 0 & \ldots & 0 & -\sqrt{\phi\psi} & s+\phi+\psi & \ldots & s+\phi+\psi & 0 & 0 \\
. & . & . & \ldots & . & . & . & \ldots & . & . & . \\
. & . & . & \ldots & . & . & . & \ldots & . & . & . \\
. & . & . & \ldots & . & . & . & \ldots & . & . & . \\
N-2 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & s+\phi+\psi & -\sqrt{\phi\psi} & 0 \\
N-1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & -\sqrt{\phi\psi} & s+\phi+\psi & -\sqrt{\phi\psi} \\
N & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & -\sqrt{\phi\psi} & s+\square+\phi
\end{array}
$$

$(N+ M) \times (N + M$

if we expand $|D (s)|$ it will be a polynomial of degree (N+M). One may note that the roots of $|D(s)|$ are the negatives of the eigen values of matrix D(0).

Observe that D(0) is a positive definite symmetric tri-diagonal matrix therefore its eigen values are real, positive and distinct, Kijima (1992). Hence the roots of the polynomial |A(s)| are real, negative and distinct (one root being zero).

Let $□_i$ (i =0, 1, 2, ..., N+M+1) be the roots of |A(s)| with ($□_υ$=$□_{–M}$= 0) then.

$$|A(s)| = s \prod_{i=1}^{N+M+1} (s - □_i)$$

and Hence $P_z(N) = \dfrac{|A_{N+M+1}(s)|}{s \prod_{i=1}^{N+M+1} (s-\alpha_i)}$ ....(12)

$□_{-M} = □_o$

$□_{-M+1} = □_1$

$□_{-M+2} = □_2$

.

.

.

$□_{N-1} = □_{N+M}$

$□_{N} = □_{N+M}$

$□_{N} = □_{N+Ml}$

Resolving eq^n (12) into partial fractions and replacing s by $\left(\dfrac{1-z}{z}\right)$ and using initial condition and comparing coefficient of z^m, we have.

$$P_m(N) = (-1)^{N+M} (□)\,(\phi)^{N+M-1}\left[\prod_{i=l}^{N+M+1}(-\alpha_i)^{-1} + \prod_{i=l}^{N+M+1}(\alpha_i)^{-1}\left(\prod_{j=1,\neq i}^{N+M+1}(\alpha_i-\alpha_j)\right)^{-1}(1+a_j)^m\right]$$

....(13)

However, if one is interested in finding the value of $P_m(N)$ under arbitrary initial condition, one may obtain the probability $P_m(N)$ as

$$P_m(N) = \prod_{j=0}^{M+N}\frac{\phi}{(-\alpha_{j+l})} + (\phi)^{N+M}\sum_{j=1}^{N+M+1} a_j(1+\alpha_j)^m \quad ....(14)$$

where

$$a_j = \frac{D_i(a_j)}{a_j \prod_{i=1,\neq J}^{N+M+1}(\alpha_j-\alpha_i)}$$

and $$P_m(N) = \prod_{j=0}^{N+M}\frac{\phi}{(-\alpha_{j+l})} + \prod_{j=l}^{N+M+1}\alpha_j(1+\alpha_j)^m \quad \text{if } i = N+M+1 \quad ....(15)$$

Where $D_i(s)$ being the determinant obtained by the top left (i×i) square matrix formed from A(s) such that $|A(s)| = D_{N+M+2}$ (s). $\square_N = \square_{N+M+1} = 0$, D_i (s) is obtained by the following recursive relation which can be easily obtained because of tridiagonal matrix of |A(s)|.

Assume D_o (s) = 1, D_1 (s) = s + □ , b_m = 0 and $\square_N$= 0

$$D_i (s) = (s+\phi+\psi)D_{i-1}(s) - \phi_{i-2}\ \psi_{i-1}\ D_{i-2} (s)\ ;\ z \leq i \leq N+M$$

It may be noted that we find the eigen values of the matrix D(0) and roots are negatives of the eigen values. The accuracy of the roots has been verified against the request in algebra viz. sum of the negatives of the roots is equal to the sum of the elements on the principal diagonal of A(0). The routines of the package are quite efficient and produce result to a high degree of accuracy even when the matrix size is greater than so.

Since $(1+\square_i)$ m □ 0 asm □ ∞ the steady state distribution is given by

$$P_m(N) = \prod_{j=0}^{N+M} \frac{\phi}{(-\alpha_{j+1})} \qquad ....(16)$$

5. Continuous Case

Letting □ = □ (□) + 0 (□) ; □ = □ (□) + 0 (□)

Taking m = t and m+1 = t + □ , in the difference differential equations one can transform the difference-differential equation in m to differential equation in t. We can then proceed to get the continuous time solutions of the transformed equations.

Alternatively, one can change the root equation and then get the continuous time solution form the final discrete time solutions. Proceeding this way the roots $\square_i$ of |A(s)| are transformed to $\square_i$□ . It is than easy to see that $(1+\square_i\square)^m$ tends to $e^{\square}i^{+1}$ in continuous time, where t is divided into m such interval each of length □ such thatt = m□ .

Now treating □ and □ as the arrival rate of passenger and arrival rate of taxi respectively. One can get the transient solution of the continuous time model. Note that the root of |A(s)| were formed in s = $\left(\frac{1-z}{z}\right)$ Parameter s in continuous case may then be treated as the transform parameter.

Now the R.H.S. of matrix equation will have l (one) in the i^{th} place instead of $\left(\frac{1}{z}\right)$.

The analogy gives the fact that discrete time model discussed in this paper are more general and correspondingly provide results in continuous time with a real case.

$$P_m(N) = (-1)^{N+M}\square\ (\phi)^{N+M-1} \left[\left\{ \prod_{i=1}^{N+M+1} (-\alpha_i) \right\}^{-1} + \prod_{i=1}^{N+M+1} \left\{ \prod_{j=1,\ j\neq i}^{N+M+1} (\alpha_i - \alpha_j)^{-1} \right\} e^{\alpha_i t} \right]$$

....(17)

6. Conclusion

A closed form transient solution for the double ended queuing system in discrete time has been obtained. It is also further shown that the corresponding results in continuous time can be obtained.

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